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Cluster structure near the percolation threshold

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Abstract. We derive exact relations that allow us to describe unambiguously and quantitatively the structure of clusters near the percolation threshold p_c. In particular, we prove the relations $p(\mathrm{d}p_{ij}/\mathrm{d}p) = \langle \lambda_{ij} \rangle$ where p is the bond density, p_{ij} is the pair connectedness function and $\langle \lambda_{ij} \rangle$ is the average number of cutting bonds between i and j. These bonds have the property such that if one is cut, i and j are no longer connected. From this relation it follows that the average number of cutting bonds between two points separated by a distance of the order of the connectedness length ξ, diverges as $|p-p_c|^{-1}$. The remaining (multiply connected) bonds in the percolating backbone, which lump together in 'blobs', diverge with a dimensionality-dependent exponent. In the light of these results, the backbone of the infinite cluster near p_c is better described by a 'nodes, links and blobs' picture, rather than the simplified Skal–Shklovskii–de Gennes 'nodes and links' model.

We also show that in the cell renormalisation group of Reynolds *et al* the 'thermal' eigenvalue is simply related to the average number of cutting bonds in the spanning cluster. Finally we discuss a percolation model in which the 'blobs' can be controlled by varying a parameter, and study the influence on the critical exponents.

1. Introduction

Percolation theory has received much attention in the last decade (see e.g. the recent reviews Stauffer (1979), Essam (1980)). Nevertheless a complete knowledge of the cluster structure is still missing. This is crucial in understanding much of the properties of random systems, such as dilute ferromagnets, random resistor networks and gels. Here we will consider bond percolation. In particular, three distinct pictures have been proposed.

(a) '*Nodes and links*' picture. Skal and Shklovskii (1975) and independently de Gennes (1976) proposed the well known 'nodes and links' model for the backbone of the infinite cluster above the percolation threshold p_c, obtained by eliminating dangling ends. In this picture the backbone is a superlattice made of nodes separated by a distance of the order of the connectedness length ξ, connected by macrobonds (figure 1(*a*)). The macrobonds are made of L links where $L \sim |p-p_c|^{-\zeta}$, p is the density of bonds, and it was argued that $\zeta = 1$. Just below p_c the backbone of the very large cluster, with linear dimension ξ, the incipient infinite cluster (IIC), is also believed to have the same structure as the macrobonds (figure 1(*b*)). This model has the nice property of being very simple to handle, but it does not accurately represent cluster structure for low dimensionality d. In fact, it predicts the same crossover

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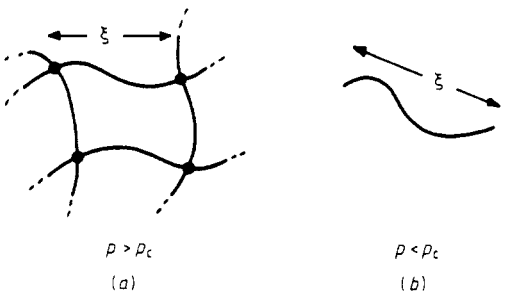

Figure 1. Skal–Shklovskii–de Gennes 'nodes and links' model of the backbone of the infinite cluster above p_c (*a*) and the incipient infinite cluster below p_c (*b*).

exponent $\phi_I = \phi_H = \zeta$ for the dilute Ising and Heisenberg systems at the percolation threshold, contrary to experimental results (Birgeneau *et al* 1980, Cowley *et al* 1980a, b). It also predicts the conductivity exponent $t = (d-2) + \zeta_R$ and the backbone order parameter exponents $\beta' = \nu d - \zeta_B$ with $\zeta_R = \zeta_B = \zeta$, which is inconsistent with the available data which require $\zeta_R \neq \zeta_B$ for low dimensionality.

(b) '*Nodes, links, and blobs*' picture. Stanley (1977) made the observation that in every configuration the bonds in the backbone of the IIC below p_c can be partitioned in two categories: the links (also called cutting bonds), such that if one is cut, the cluster breaks into two parts, and the remaining multiply connected bonds which lump together in 'blobs'. Recently this picture was put on a more quantitative basis through the study of the dilute Heisenberg and Ising systems near p_c. It was pointed out that each type of bond plays an important role (Coniglio 1981a) and it was also suggested (Coniglio 1981b) that the infinite cluster above p_c can be described as nodes connected by links and blobs.

(c) '*Sierpinski gasket backbone*' model. Gefen *et al* (1981) have introduced an alternative model that represents the opposite extreme of the nodes and links picture. This is a fractal model for the backbone of the infinite cluster at p_c. It has a self-similar structure, and *only* multiply connected bonds are present. Thus in the 'nodes and links' model the blobs are neglected, while in the Sierpinski gasket model the links are neglected. A great advantage of this model is that, like some hierarchical models in thermal phase transitions (Berker and Ostlund 1979), it can be solved exactly. It also gives good results in low dimensions for the backbone and the conductivity exponents but it fails to predict the correct value for the dilute Ising crossover exponent ϕ_I for any d.

In this paper we derive exact relations which allow us to describe unambiguously and quantitatively the structure of clusters near p_c. In particular, we show that both links and blobs are important, thus supporting picture (b). More quantitatively, we find that the number of links L between two points separated by a distance of the order ξ diverges as $L \sim |p - p_c|^{-1}$ as $p \to p_c^-$ for any lattice and dimensionality d. This result has been confirmed on the square lattice by Pike and Stanley (1981) using Monte Carlo methods.

Using this result, and the numerical information (see table 1) that the number of backbone bonds between two points separated by a distance of the order of ξ diverges with an exponent $\zeta_B > 1$ for $2 \leq d < 6$, it follows that the number of bonds in the blobs also diverges with the same exponent ζ_B. Therefore below p_c the backbone of the

Table 1.

d	1	2	3	4	5	6 [f]
β [a]	0	5/36	0.45	0.58	0.76	1
β' [b]	0	0.5–0.6	0.8–1.0	1.0–1.2		2
ν [a]	1	4/3	0.88	0.7	0.6	1/2
ζ_R [c]	1	1.43 + 0.02	1.12 + 0.02	1.05 + 0.02	1.02 + 0.02	1
ζ_{min} [d]	1	1.49 + 0.01 [e]				1
ζ_{SAW} [e]	1	1.73	1.32	1.21	1.15	1
$\zeta_B = \nu d - \beta'$		2.16–2.06	1.84–1.64	1.8–1.6		1

[a] From best estimates of Stauffer (1979) and Stauffer *et al* (1982).
[b] Kirkpatrick (1978), and for $d = 2$ also, Shlifer *et al* (1979).
[c] Fisch and Harris (1978).
[d] Pike and Stanley (1981).
[e] This exponent has been evaluated via $\zeta_{SAW} = \nu/\nu^{p_c}_{SAW}$ where $\nu^{p_c}_{SAW} = 3/(2 + d_F)$ is the Flory formula for the SAW exponent extended by Kremer (1981) to the percolating cluster with fractal dimensionality $d_F = d - \beta/\nu$.
[f] Exact results on the Cayley tree.

IIC on a length scale of the order of ξ looks like a quasi-one-dimensional chain made of links and very large blobs.

On the other hand, for $d > 1$ we argue that the number of cutting bonds on the backbone of the IIC, between two points separated by a distance much smaller than ξ, tends to zero as $|p - p_c|^{-\alpha}$ where α is the 'specific heat' exponent, which in percolation is known to be negative. Therefore for $p = p_c$, or equivalently for length scales much smaller than ξ according to our non-rigorous argument, there are no cutting bonds between points on the backbone of the IIC. However we show that internal links are still present (figure 2(*c*)).

The remainder of this paper is organised in the following way. In § 2 we give the basic relations which support the 'nodes, links and blobs' picture below p_c. In § 3 we give relations which suggest that the blobs are made of quasi-one-dimensional chains of links and blobs with a self-similar structure (figure 2(*c*)). In § 4 we show that for

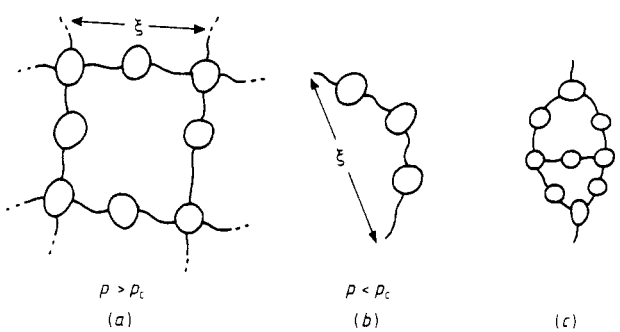

Figure 2. 'Nodes, links, and blobs' model. A modified version of the 'nodes and links' picture of the backbone of the infinite cluster above p_c. (*a*) Nodes separated by a distance ξ are connected by links and blobs. Note that a node may be a blob. (*b*) Backbone of the incipient infinite cluster below p_c made of links and blobs. (*c*) Structure of a blob: points separated by a distance b, such that $a \ll b \leq \xi$ where a is the lattice space, are connected by chains made of links and 'blobs' in a self-similar structure.

simplicity one can substitute the backbone of the IIC with an effective one-dimensional chain, like in the nodes and links model, provided that its length depends on the quantity under consideration, such as resistivity, shortest path, self-avoiding walk on a dilute lattice, size of the backbone. We show that all these lengths satisfy a hierarchy of inequalities. Different exponents are associated with the divergence of these lengths, which are ≥ 1. An analysis of the available numerical data shows that these exponents take their maximum value at $d = 2$ where the blobs are most important and tend to 1 monotonically as $d \to 6$, where the influence of the blobs vanishes.

In § 5, the previous analysis on the cluster structure is extended to $p > p_c$ and the nodes, links and blobs picture is confirmed.

In § 6, we show in the context of the cell renormalisation group theory (Reynolds *et al* 1978, 1980) that the cutting bonds play a crucial role in driving the percolation transition. We also show that the 'thermal' eigenvalue is simply related to the average number of cutting bonds in the spanning cluster.

In § 7 we discuss the influence of the blobs on the critical exponents. To do so we consider a variation of the percolation model which has been introduced in the context of polymer gelation and vulcanisation (Coniglio and Daoud 1979, Ord and Whittington 1982). In this model, besides the usual unrestricted bond, another bond is introduced with restricted valence 2. This has the property of forming chains and therefore produces the effect of reducing the size of the 'blobs'. This can be done until the 'blobs' disappear and crossover to self-avoiding walk behaviour is observed.

The conclusions are given in § 8. The proofs of the new relations presented in this paper are given in the appendix. The essential results given here have been reported briefly in previous publications (Coniglio 1981a, b).

2. Cluster structure below p_c

Consider bond percolation on a regular lattice in d dimensions. A basic quantity is the pair connectedness function p_{ij}, defined as the probability that i and j belong to the same cluster. For $p \leq p_c$ the connectedness length ξ can be defined as (Essam 1980)

$$\xi^2 = \Big(\sum_j r_{ij}^2 p_{ij}\Big) \Big/ \Big(\sum_j p_{ij}\Big), \tag{1}$$

where r_{ij} is the geometric ('crow-flying') distance between i and j. The mean cluster size S is given by

$$S = \sum_j p_{ij}. \tag{2}$$

These quantities diverge as $p \to p_c$ as $\xi \sim \varepsilon^{-\nu}$ and $S \sim \varepsilon^{-\gamma}$ where $\varepsilon = |p - p_c|/p_c$.

In order to obtain more information about the cluster structure, we distinguish several types of bonds according to the role they play in connectivity. For example, the backbone bonds between i and j are the bonds that belong to at least one self-avoiding walk between i and j. The remaining bonds in the cluster are dangling bonds. The backbone bonds between i and j are made of singly connected or cutting bonds (red bonds in the terminology of Pike and Stanley (1981)), which have the property that if one of these bonds is cut, i and j are no longer connected, and the remaining bonds are multiply connected (figure 3).

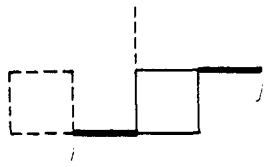

Figure 3. Example of a ten-bond cluster. Between sites i and j we have (a) cutting bonds $\lambda_{ij} = 2$ (bold lines), (b) resistance $\rho_{ij} = 3$, (c) shortest path $\mu_{ij} = 4$, (d) average number of bonds among all self-avoiding walks $\bar{\tau}_{ij} = 4$, (e) backbone bonds $\beta_{ij} = 6$ (full lines), and (f) dangling ends = 5 (broken lines).

We note that the backbone defined here is always related to two sites. Accordingly the backbone of the IIC below p_c is defined as the backbone between two sites separated by a distance ξ, in a 'typical' cluster of linear dimension ξ, while the backbone of the infinite cluster above p_c is better defined as that part made of sites which are biconnected, such that any two sites can be connected via two independent paths made of different bonds (Kirkpatrick 1978, Shlifer *et al* 1979).

For every configuration of bonds we define λ_{ij} as the number of cutting bonds between i and j and the average over all the bond configurations denoted by $\langle\lambda_{ij}\rangle$. For $p \leqslant p_c$, analogously to (1), we define a quantity L

$$L = \sum_j \langle\lambda_{ij}\rangle / \sum_j p_{ij} \tag{3}$$

which is roughly the average number of cutting bonds between sites separated by a distance of the connectedness length ξ (figure 2(b)) or the number of cutting bonds in the backbone of the IIC.

In the appendix we will show that for any lattice and dimensionality d, the following relation holds between $\langle\lambda_{ij}\rangle$ and the pair connectedness function p_{ij}:

$$p \, \mathrm{d}p_{ij}/\mathrm{d}p = \langle\lambda_{ij}\rangle. \tag{4}$$

This new relation is one of the basic results of this paper. The physical meaning of equation (4) can be understood in the following way. Suppose one decreases the density of bonds by an amount $\mathrm{d}p$. The consequent decrease in the pair connectedness function $\mathrm{d}p_{ij}$ is given by the fraction of bonds removed $(\mathrm{d}p)/p$ times the average number of cutting bonds between i and j. In fact, these are the only bonds which will reduce the connectivity due to an infinitesimal decrease of the bond probability.

In site percolation (4) is also valid where p is the density of occupied sites and $\langle\lambda_{ij}\rangle$ is the average number of articulation points (or red sites in the terminology of Pike and Stanley (1981)), which have the property that if one is removed i and j are no longer connected.

If we sum both terms in (4) and divide by $S = \Sigma_j p_{ij}$ we obtain

$$L = (p/S)\,\mathrm{d}S/\mathrm{d}p \sim \varepsilon^{-1} \tag{5}$$

which shows that the number of cutting bonds in the IIC becomes a critical quantity at p_c. The result given in equation (5) was already predicted in a previous paper (Coniglio 1981a, b) using a renormalisation group procedure, and confirmed by Monte Carlo methods for the square lattice by Pike and Stanley (1981).

The quantity L plays an important role in the quenched dilute s-state Potts model near $p = p_c$ and $T = 0$. In fact, it has been shown (Coniglio 1981a, b) that the crossover

exponent ϕ for such a model coincides with the critical exponent describing the divergence of L, namely $\phi = 1$ for any lattice and d. This is in agreement with the ε expansion (Stephen and Grest 1977, Wallace and Young 1978) and experimental results (Cowley *et al* 1980a, b) on Ising systems. This also explains why the backbone model of Gefen *et al* fails to give the right crossover exponent, due to the absence of links in the model. The importance of the links in determining the crossover exponent ϕ has also been pointed out by Ziman (1979).

Finally, from the scaling form of the pair connectedness (Essam 1980)

$$p_{ij} = (1/r_{ij}^{d-2+\eta}) f(r_{ij}/\xi) \tag{6}$$

and from equation (4) follows

$$\langle \lambda_{ij} \rangle = (1/r_{ij}^{d-2+\eta-1/\nu}) f_1(r_{ij}/\xi) \tag{7}$$

where $f_1(x) = x(\mathrm{d}f/\mathrm{d}x) r_{ij}^{-1/\nu} (\mathrm{d} \log \xi / \mathrm{d}p)$ with $x = r_{ij}/\xi$. It is convenient to define $L_{ij} = \langle \lambda_{ij} \rangle / p_{ij}$, the average number of cutting bonds between i and j under the condition of being connected. The scaling form for L_{ij} is

$$L_{ij} = r_{ij}^{1/\nu} f_2(r_{ij}/\xi), \tag{8}$$

with $f_2(x) = f_1(x)/f(x)$. From (8), putting $r_{ij} = \xi$, it follows that $L_{ij} \sim \varepsilon^{-1}$ which is another way of deriving equation (5). However (8) is more general and will be used later in the limit $r_{ij}/\xi \ll 1$.

From equation (5) alone it is not possible to deduce whether the 'blobs' are relevant or not. However, using other numerical data it is easy to realise that these blobs cannot be neglected for low dimensions. In fact, if the blobs were not present, the backbone of the IIC would have been made only of a linear chain made of L steps, with the obvious inequality $L \geqslant \xi$, since the end-to-end distance ξ is smaller than any other path. From (5) it would have followed that $\nu \leqslant 1$ which contradicts the well established result $\nu = \frac{4}{3}$ in two dimensions (den Nijs 1979, Reynolds *et al* 1978, 1980, Eschbach *et al* 1981, Blöte *et al* 1981, Nienhuis 1982). Therefore the 'blobs' must necessarily be present in two dimensions so that L can be smaller than ξ. In higher dimensions $\nu < 1$, therefore we apply another argument to show that the 'blobs' are relevant.

Below p_c the total number of backbone bonds L_{BB} on the IIC can be defined as

$$L_{BB} = \sum_j \langle \beta_{ij} \rangle \Big/ \sum_j p_{ij}$$

where $\langle \beta_{ij} \rangle$ is the average number of backbone bonds between i and j. Clearly $L_{BB} = L + L_B$ where L is the number of cutting bonds and L_B is the total number of bonds in the blobs. From the available numerical data (see table 1) $L_{BB} \sim \varepsilon^{-\zeta_B}$ with $\zeta_B > 1$ for $2 \leqslant d < 6$. Using the exact result $L \sim \varepsilon^{-1}$ it follows that $L_B \sim L_{BB} \sim \varepsilon^{-\zeta_B}$ for such values of d. In other words, near the phase transition nearly all the backbone bonds belong to blobs and are not cutting bonds.

So far we have studied the properties of the IIC for length scales of the order of ξ. In particular, we stress that the average number of cutting bonds L in equation (6) is evaluated between two connected points separated by a distance ξ. As $p \to p_c$ the distance between the two points diverges. We want to consider now the case in which the two points on the backbone are separated by a fixed distance $r_{ij} \ll \xi$ in the limit $p \to p_c$. Since $\lim_{p \to p_c} (L/L_{BB}) = 0$ we expect that the number of cutting bonds L_{ij}^{BB} between these two points goes to zero as $p \to p_c$.

These results give the following picture. Suppose $p < p_c$, and two points i and j in the same cluster, separated by a distance of the order of ξ, are connected by a quasi-one-dimensional chain of blobs and cutting bonds. If $p \to p_c$ the cluster will grow and the chain connecting the fixed points i and j will eventually become part of a blob (see also Pike and Stanley 1981). In this nonlinear process the number of cutting bonds between i and j is expected to go to zero with a critical exponent as $p \to p_c$. To find such an exponent we consider (8) in the limit $r_{ij}/\xi \gg 1$. In such a limit we expect in analogy with thermal phase transitions (Fisher and Burford 1967, Coniglio and Marinaro 1973)

$$f(r_{ij}/\xi) - f_{ij}(0) \sim r_{ij}^{(1-\alpha)\nu^{-1}} \varepsilon^{1-\alpha} + B(r_{ij})\varepsilon + \ldots . \tag{9}$$

Here α is the 'specific heat' exponent, and the first and second term are related respectively to the singular and regular part of the 'energy'†. From (6)–(8) follows

$$L_{ij} \sim r_{ij}^{(1-\alpha)\nu^{-1}} \varepsilon^{-\alpha} + B(r_{ij}). \tag{10}$$

The singular part can be interpreted as the contribution to the average number of cutting bonds L_{ij}^{BB} coming from those configurations in which i and j are on the backbone of the IIC and goes to zero as expected, since $\alpha < 0$ for all $d > 1$. The regular part $B(r_{ij})$ may be interpreted as the contribution coming from those configurations in which i and j belong to small clusters or dangling ends. It would be interesting to verify this result by other methods.

In conclusion, the backbone of the IIC *for length scales of the order of* ξ looks like a quasi-one-dimensional chain made of cutting bonds and blobs (figure 2(*b*)). The number of cutting bonds diverges with a critical exponent 1 in any dimension, while the number of bonds in the blobs diverges with an exponent larger than unity given by the backbone exponent ζ_B for $2 \leqslant d < 6$. For $d \geqslant 6$ it is believed that the backbone can be well approximated by a Cayley tree (de Gennes 1976), therefore the blobs can be neglected. Consequently, $L_{BB} \sim L - \varepsilon^{-1}$ in agreement with numerical data (Gefen *et al* 1981). On the other hand, *for length scales much smaller than* ξ we have argued that the number of cutting bonds between backbone sites goes to zero with the exponent $-\alpha$, and therefore right at p_c for finite length scales no cutting bonds are present but only blobs. However in the next section we will give a relation which suggests that the blobs themselves are made of internal links and blobs in a self-similar structure.

3. Structure of the blobs and self similarity

In order to obtain information on the structure of the blobs, we have generalised relation (4) to pairs of biconnected points. These are points which are connected via at least two independent paths, with no common bonds. For $p \leqslant p_c$ only points within one blob have this property. The relevant relation which is proved in the appendix is

$$p \, \mathrm{d}p_{ij}^{B}/\mathrm{d}p = \langle \lambda_{ij}^{B} \rangle \tag{11}$$

where p_{ij}^{B} is the probability that i and j are biconnected, which we call the pair

† A different conjecture (Stauffer 1978) predicts for the singular part of (9) a behaviour proportional to $\varepsilon^{2\beta}$. This would imply that (10) would diverge as $\varepsilon \to 0$ for $d = 2, 3$ which is hard to expect on physical grounds.

biconnectedness function, and $\langle \lambda_{ij}^{B} \rangle$ is the average number of bonds such that if one is cut i and j are no longer biconnected. If the mean size of the blobs $S^{B} = \Sigma_j p_{ij}^{B}$ diverges, which is the case for $2 \leq d < 6$, it follows for $p \leq p_c$ that

$$L_B = \sum_j \langle \lambda_{ij}^{B} \rangle \Big/ \sum_j p_{ij}^{B} \sim \varepsilon^{-1} \tag{12}$$

where L_B is roughly the number of bonds between two points in a blob, separated by a distance of the order of ξ, such that if one is cut the two points are no longer biconnected. Equation (12) suggests that also the blobs are made of chains with links and blobs in a self-similar structure down to length scales large compared with the lattice spacing a (figure 2). Note that since these 'internal' links belong to the blobs, they are not cutting bonds. While the cutting bonds scale as $b^{1/\nu}$ only for length scales $b \sim \xi$, the internal links scale as $b^{1/\nu}$ for any length scale $a \ll b \leq \xi$. (For a discussion on the internal links, see also Gefen *et al* (1981).) The self-similarity property in the percolating cluster has been discussed by many authors (e.g. Stanley *et al* 1976, Mandelbrot 1977, Kirkpatrick 1979, Stauffer 1979, Gefen *et al* 1981, Stanley 1982).

4. Effective one-dimensional lengths associated with the IIC

In the simplified 'links and nodes' picture, the backbone of the IIC below p_c is made only of a chain of L links, therefore all the quantities such as resistivity, shortest path length and total number of bonds coincide with L. In the modified version discussed here, due to the presence of the blobs, all these quantities are different from each other. We can associate with any of these quantities an effective one-dimensional length. Although we cannot predict the divergence of these lengths as we did for L, we can construct a hierarchy of inequalities which might be a guide in interpreting the numerical data, and also useful to characterise the structure of the blobs.

Let us consider the following two-point functions expressed in scaling form for $p < p_c$ (see figure 3).

(i) Average resistance between i and j:

$$\langle \rho_{ij} \rangle = (1/r_{ij}^{d-2+\eta-\zeta_R/\nu}) f_3(r_{ij}/\xi). \tag{13a}$$

The average resistance L_R between two sites separated by a distance of order of ξ, or equivalently the resistance of the backbone of the IIC, is

$$L_R = (1/S) \sum_j \langle \rho_{ij} \rangle \sim \varepsilon^{-\zeta_R}. \tag{13b}$$

The quantities (13a) and (13b) were introduced by Fisch and Harris (1978) who calculated the critical exponent ζ_R, up to six dimensions, using low density series expansions (see also the ε expansion near six dimensions of Dasgupta *et al* (1978)). Recently ζ_R has been shown to coincide with the crossover exponent in the dilute n-vector model ($n > 1$) near p_c (Coniglio 1981a, b; see also Gefen *et al* 1981), in agreement with previous results which related dilute ferromagnets near $T = 0$ to random resistor networks (Kirkpatrick 1973, Stinchcombe 1979). This exponent seems to be larger than the corresponding exponent ζ_R defined above p_c via the conductivity exponent $t = (d-2)\nu + \zeta_R$ (see e.g. Straley 1977, Harris and Kirkpatrick 1977, Fogelholm 1980). This discrepancy may be due to corrections coming from the fact that the blobs at the nodes (figure 2) might be more dense than in the asymptotic limit $p \to p_c$ (Deutscher 1981).

(ii) The average number of bonds in the shortest path connecting i and j

$$\langle \mu_{ij} \rangle = (1/r_{ij}^{d-2+\eta-\zeta_{\min}/\nu}) f_4(r_{ij}/\xi) \tag{14a}$$

while the number of bonds $L_{\min}$ in the shortest path of the backbone of the IIC is

$$L_{\min} = (1/S) \sum_j \langle \mu_{ij} \rangle \sim \varepsilon^{-\zeta_{\min}}. \tag{14b}$$

This quantity has been investigated by Kirkpatrick (1978) and Whittington *et al* (1980) and its critical exponent for random percolation has been estimated by Pike and Stanley (1981).

(iii) The average number of bonds in the backbone between i and j is

$$\langle \beta_{ij} \rangle = (1/r_{ij}^{d-2+\eta-\zeta_B/\nu}) f_5(r_{ij}/\xi) \tag{15a}$$

while the number of bonds L_{BB} in the IIC is

$$L_{BB} = (1/S) \sum_j \langle \beta_{ij} \rangle \sim \varepsilon^{-\zeta_B}. \tag{15b}$$

The exponent ζ_B can be related to the order parameter backbone exponent $\beta' = \nu d - \zeta_B$ and has been calculated for different values of d by Kirkpatrick (1978) and for $d = 2$ by Shlifer *et al* (1979).

(iv) Finally we introduce the following quantity which is related to the statistics of the self-avoiding walk (SAW) on a dilute lattice:

$$\langle \bar{\tau}_{ij} \rangle = (1/r_{ij}^{d-2+\eta-\zeta_{SAW}/\nu}) f_6(r_{ij}/\xi) \tag{16a}$$

where $\bar{\tau}_{ij}$ is the mean number of steps in the set of the SAW between i and j in a given configuration of bonds. The mean number of steps L_{SAW} among the SAW between the extreme ends of the IIC is therefore given by

$$L_{SAW} = (1/S) \sum_j \langle \bar{\tau}_{ij} \rangle \sim \varepsilon^{-\zeta_{SAW}}. \tag{16b}$$

An analysis of the statistics of the SAW on random lattices has been recently done by Chakrabarti and Kertész (1981) and Kremer (1981) using Monte Carlo methods, and by Derrida (1982) with the transfer matrix approach. Kremer, on the base of his numerical result, suggested that a good approximation for the self-avoiding walk exponent $\nu_{SAW}^{p_c} = \nu/\zeta_{SAW}$ on a dilute lattice at the percolation threshold might be given by the Flory formula

$$\nu_{SAW}^{p_c} = 3/(d_F + 2) \tag{17}$$

where $d_F = d - \beta/\nu$ is the fractal dimensionality of the percolating cluster at p_c, and β is the order parameter percolation exponent.

The physical meaning of L_R is that the resistance of the backbone of the IIC is equivalent to a one-dimensional chain made of L_R steps. The same interpretation may be given to the other quantities $L_{\min}$, L_{SAW}, L_{BB}. Therefore the backbone of the IIC can be associated with a one-dimensional chain like in the 'nodes and links' picture, provided that different lengths are used in correspondence of different quantities.

Since $\lambda_{ij} \leqslant \rho_{ij} \leqslant \mu_{ij} \leqslant \bar{\tau}_{ij} \leqslant \beta_{ij}$ as figure 3 clearly shows, from (9), (11), (13), (16) it follows that

$$L \leqslant L_R \leqslant L_{\min} \leqslant L_{SAW} \leqslant L_{BB}, \tag{18}$$

$$\xi \leqslant L_{\min} \leqslant L_{SAW} \leqslant L_{BB}, \tag{19}$$

from which the following exponent inequalities are obtained:

$$1 \leqslant \zeta_R \leqslant \zeta_{min} \leqslant \zeta_{SAW} \leqslant \zeta_B, \tag{20}$$

$$\nu \leqslant \zeta_{min} \leqslant \zeta_{SAW} \leqslant \zeta_B. \tag{21}$$

If the blobs were irrelevant all the lengths in (18) would coincide with L, and all the exponents in (20) would be equal to 1. This is the case for $d = 1$ and $d \geqslant 6$, where mean field theory (without loops) is valid†.

For $2 \leqslant d \leqslant 6$, the difference of the various exponents from 1 gives a measure of the relevance of the blobs. In table 1 and in figure 4 we have given the estimated exponents as a function of the dimensionality. In $d = 2$ the exponents have their largest values. For $d = 3$, ζ_R is almost 1 while the backbone exponent ζ_B is still quite large. This means that the blobs are still present but do not strongly influence the resistance.

Finally we note that the exponents ζ_R and ζ_{min} are very close for $d = 2$. This is due to the fact that the shortest path gives the dominant contribution to the resistance. We expect that the two exponents should become even closer for higher d where the blobs are less important.

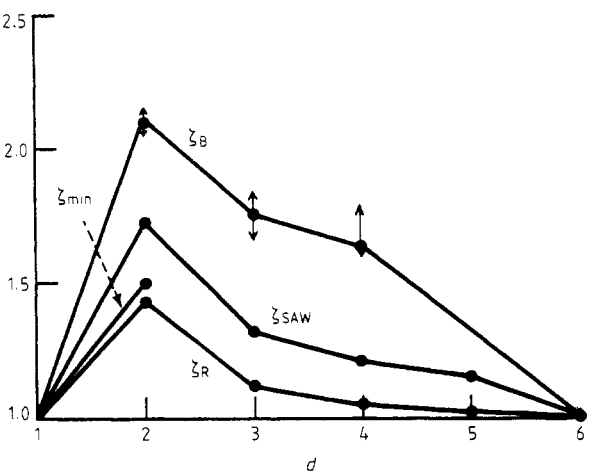

Figure 4. Critical exponents from table 1 plotted as function of d. Note that $1 \leqslant \zeta_R \leqslant \zeta_{min} \leqslant \zeta_{SAW} \leqslant \zeta_B$ as predicted from equation (20). As is shown in the text, these exponents are equal to 1 if there are no 'blobs' in the backbone of the incipient infinite cluster. In fact we note here that they are equal to 1 for $d = 1$, where it is impossible to have blobs, take their maximum value at $d = 2$ and tend again to 1 as d approaches 6, implying that the influence of the 'blobs' decreases and vanishes at $d = 6$ where mean field theory (no loops) takes over. The lines are intended as a guide for the eye. The error bars have been indicated with double arrows, when larger than the size of the dots.

5. Cluster structure above p_c

Consider a d-dimensional hypercube of linear dimension. The probability R that one face is connected to the opposite in the limit of large b tends to a step function (figure 5), and is related to the connectedness length exponent ν by finite size scaling via

† Note that for $d = 1$, ξ also coincides with all the lengths and $\nu = 1$, while for $d = 6$, $\nu = \frac{1}{2}$.

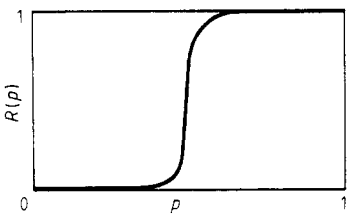

Figure 5. Schematic behaviour of R: the probability of getting across from one face to the opposite of a given cell of linear dimension b (Reynolds *et al* 1978, 1980). From relation (23), $p(\mathrm{d}R/\mathrm{d}p) = \langle\lambda\rangle$ where $\langle\lambda\rangle$ is the average number of cutting bonds between the opposite faces, it follows that $\langle\lambda\rangle$ is practically zero everywhere except in a narrow region $\Delta p \sim b^{-1/\nu}$ around p_c where it suddenly reaches a value $\sim b^{-1/\nu}$.

(Levinshtein *et al* 1976, Reynolds *et al* 1978, 1980, Kirkpatrick 1978)

$$p\ \mathrm{d}R/\mathrm{d}p|_{p=p^*} = Ab^{1/\nu}, \tag{22}$$

where A is a constant and p^* is a suitable value of p (usually chosen as the value for which $\mathrm{d}R/\mathrm{d}p$ has a maximum), which scales as $p^* - p_c \sim b^{-1/\nu}$ where p_c is the percolation threshold of the infinite system. The following relation has been proved in the appendix:

$$p\ \mathrm{d}R/\mathrm{d}p = \langle\lambda\rangle, \tag{23}$$

where $\langle\lambda\rangle$ is the average number of cutting bonds between the two opposite faces of the hypercube, such that if one is cut one face is no longer connected to the other.

For the special case $b = \xi(p)$, from (22) and (23) follows

$$\langle\lambda\rangle = A|p - p_c|^{-1}. \tag{24}$$

Relations (23) and (24) have also been proved independently by Kertesz (private communication). For $p > p_c$ in the above relation $\langle\lambda\rangle$ can be interpreted as the number of cutting bonds between two nodes in the infinite cluster separated by a distance of the order of ξ (figure 6). Therefore, using the same argument as for $p < p_c$, it follows from (24) that the backbone of the infinite cluster can be considered as a superlattice made of *nodes* separated by a distance ξ, connected by *links* and *blobs*, with the same self-similar structure as the IIC below p_c (figure 2).

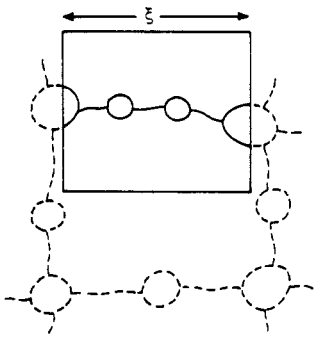

Figure 6. A cell of size $b \sim \xi$ is shown, along with the backbone of a spanning cluster (full lines). If $p > p_c$, can be considered part of the backbone of the infinite cluster (broken lines).

Assuming the 'links and nodes' model, Skal and Shklovskii (1975) argued and de Gennes (1976) postulated that the number of links connecting the nodes diverges as (24).

Information about the structure of the dangling ends of the infinite cluster can be obtained from equation (4) for $p > p_c$:

$$p \, \mathrm{d}p_{ij}/\mathrm{d}p = \langle \lambda_{ij} \rangle. \tag{25}$$

Note that p_{ij} is the probability that i and j are connected via a finite or infinite cluster. Taking the limit $|i-j| \to \infty$ and dividing both terms in (25) by $P^2 = \lim_{|i-j|\to\infty} p_{ij}$, we have

$$(2p/P) \, \mathrm{d}P/\mathrm{d}p = \langle \lambda_\infty \rangle / P^2 \sim \varepsilon^{-1}. \tag{26}$$

P is the percolation order parameter and $\langle \lambda_\infty \rangle / P^2$ is the average number of cutting bonds between two infinitely distant sites supposedly connected by the infinite cluster. Consequently these are bonds in the dangling ends of the infinite cluster. Equation (26) suggests that the dangling ends of the infinite cluster also have a structure similar to the IIC below p_c.

6. Role of the cutting bonds in the percolation transition

Although the cutting bonds are a minority compared with the bonds in the blobs, nevertheless they play an essential role in driving the percolation transition. If we look at the behaviour of the probability R of 'getting across' a large cell (figure 5), we note that in both the percolative and non-percolative phase $\mathrm{d}R/\mathrm{d}p$ is practically zero and from (23) no cutting bonds are present. Only in the transition regime does R change abruptly and a cascade of cutting bonds is produced. This suggests that the transition to the percolative regime occurs by means of a coalescence of clusters via cutting bonds.

We also note from (22) and (23) that

$$\nu^{-1} = \lim_{b\to\infty} \frac{\ln \langle \lambda \rangle_{\max}}{\ln b}, \tag{27}$$

where $\langle \lambda \rangle_{\max}$ is the average number of cutting bonds calculated at $p = p^*$ where it assumes its maximum value. The above relation provides a direct geometrical interpretation of the connectedness length exponent ν and a direct method for its computation using the large cell renormalisation group mostly developed by Reynolds *et al* (1978, 1980).

7. A percolation model in which the 'blobs' can be inhibited

In this section we want to discuss a percolation model in which the blobs can be reduced in size by changing a parameter and study the effect on the critical exponents. In this model two types of bonds, A and B, may be active. The A bonds with concentration c_A have a restricted valence 2, namely no more than two A bonds can come out of the same vertex. These bonds have the property of forming long chains. The B bonds with concentration c_B have no restrictions as in the usual percolation problem. Clusters are made of sites connected by either type of bond. Two limiting cases are well known: $c_A \neq 0$, $c_B = 0$ describes a system of SAW chains; $c_A = 0$, $c_B \neq 0$

give the usual bond percolation. The general case $c_A \neq 0$, $c_B \neq 0$ has been studied by Coniglio and Daoud (1979) and more recently by Ord and Whittington (1982) in the context of polymer gelation and vulcanisation. Here the interest in this model stems from the fact that the presence of the A bonds inhibits the formation of the blobs.

In the renormalisation group approach of Coniglio and Daoud (1979), it is found that the critical exponents are the same as for the usual random bond percolation problem, except for special values of the parameters where crossover to SAW exponents is observed. This SAW regime is characterised by long chains made of A bonds and few B bonds acting as crosslinks among the chains. This result can be understood in the context of the 'nodes, links and blobs' picture. As the concentration of the B bonds increases, the size of the blobs is reduced until they disappear completely and a crossover to the nodes and links picture occurs, in which nodes separated by a distance ξ are connected by SAW chains.

Recently, experiments have been done on gels obtained from a solution of mono- and bisacrylamide (Gupta and Bansil 1981). The monoacrylamide is a two-valence monomer that tends to form chains (like the A bond), while the bisacrylamide acts as a crosslink (like the B bond). The experiment showed that by increasing the relative concentration of mono- with respect to the bisacrylamide the gel changed from an opalescent phase to a clear phase. This could be explained by a transition from a nodes, links and blobs structure (the blobs being responsible for the opalescence) to a nodes and links structure.

8. Conclusion

In conclusion, we have derived exact relations in percolation theory (see equations (4), (5), (10), (23) and (26)) which have allowed us to describe quantitatively the structure of clusters near the percolation threshold. Our analysis shows that both links and blobs (Stanley 1977) play an important role in the description of the backbone of the IIC, strongly supporting the nodes, links and blobs picture. In fact if the blobs were neglected, as in the nodes and links picture, from the exact result (4) it would have followed that $\nu \leqslant 1$ for any dimension d, contrary to the 'exact' result $\nu = \frac{4}{3}$ for $d = 2$. Moreover, an entire set of critical exponents would have been equal to 1, contrary to numerical data (see table 1). On the other hand, from equations (5) and (11) it follows that for all d the links cannot be neglected as in the Sierpinski gasket backbone model of Gefen *et al* (1981). Therefore one should try to generalise this model to include links. Although the average number of links is much smaller than the bonds in the blobs for $2 \leqslant d \leqslant 6$, which is the reason why they are difficult to see in a computer experiment, they play an important role in driving the percolation transition (see e.g. figure 5). Moreover, they determine the value of the crossover exponent, which is equal to 1 in the quenched dilute Ising model and more in general in the quenched dilute s-state Potts model at the percolation threshold. Finally, they also determine the value of the connectedness length exponent ν (equation (27)).

It is useful to compare the behaviour of the different types of bonds in a volume of size b at p_c:

(i) (total number of bonds) $\sim b^d$;

(ii) (number of bonds in the IIC) $\sim$ (number of dangling bonds) $\equiv L_D \sim b^{d-\beta/\nu}$;

(iii) (number of backbone bonds) $\sim$ (number of bonds in the blobs) $\equiv L_{BB} \sim b^{d-\beta'/\nu}$;

(iv) (number of cutting bonds) $\equiv L \sim b^{1/\nu} = b^{d-(1-\alpha)/\nu}$.

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Appendix

In order to derive equations (4), (11) and (23) consider the following bond-dilute percolation problem on a general lattice. Suppose that we have two kinds of bonds, A and B. Distribute at random the A bonds with occupation probability p_A and the B bonds with occupation probability p_B. The B bonds are allowed to fall on top of the A bonds. Consider active a link which is occupied by both an A and a B bond. The probability p for such a link to be active is given by

$$p = p_B p_A. \tag{A1}$$

Therefore if $p_{ij}(p_B)$ is the pair connectedness function for the usual percolation problem relative to the B bonds only, the pair connectedness function for this bond-dilute percolation problem is given by $p_{ij}(p_B p_A)$. This bond-dilute percolation can also be obtained from the quenched or annealed Q-state Potts model in the limit $Q \to 1$ (Yeomans and Stinchcombe 1980, Coniglio 1981a, b). We calculate the pair connectedness function $p_{ij}^{\Lambda}(p_B p_A)$ explicitly first in a finite volume Λ and then take the thermodynamic limit $p_{ij}(p_B p_A) = \lim_{\Lambda \to \infty} p_{ij}^{\Lambda}(p_B p_A)$. For convenience in what follows we will omit the superscript Λ:

$$p_{ij}(p_B p_A) = \sum_C T_{ij}(p_A; C)\Pi_{ij}(p_B; C) \tag{A2}$$

where $T_{ij}(p_A; C)$ is the probability that the configuration C of A bonds in which i and j are connected occurs. $\Pi_{ij}(p_B; C)$ is the probability that i and j are connected by B bonds in the sublattice made of active A bonds in the configuration C. The sum is over all configurations C containing i and j. We calculate now the probability that i and j are not connected by B bonds, $\tilde{\Pi}_{ij}(p_B; C) = 1 - \Pi_{ij}(p_B; C)$, to the first order in $q_B = 1 - p_B$,

$$\tilde{\Pi}_{ij}(p_B; C) = \lambda_{ij} q_B + O(q_B^2), \tag{A3}$$

where λ_{ij} is the number of cutting bonds between i and j in the configuration C and $O(q_B^2)$ are higher-order terms in q_B. The linear term in q_B is explained because there are exactly λ_{ij} configurations such that i and j are not connected with only one B bond missing. From (A2) and (A3) we have

$$p_{ij}(p_B p_A) = p_{ij}(p_A) - \sum_C T_{ij}(p_A; C)[q_B \lambda_{ij} + O(q_B^2)]. \tag{A4}$$

Taking the derivative of both terms with respect to p_B calculated at $p_B = 1$, we have

$$p \, dp_{ij}/dp = \langle \lambda_{ij} \rangle \tag{A5}$$

where $p = p_A$. This relation has been obtained for a finite system. If we take the

thermodynamic limit it is valid also for an infinite system. In this case p_{ij} is the probability that i and j are connected via a finite or infinite cluster.

This relation is a particular case of the following general relation valid for any system either finite or infinite in any dimension d:

$$p\ \mathrm{d}p_{CD}/\mathrm{d}p = \langle \lambda_{CD} \rangle \tag{A6}$$

where C and D are given sets of points, p_{CD} is the probability that at least one point in C is connected to at least one point in D and $\langle \lambda_{CD} \rangle$ is the average number of cutting bonds between C and D such that if one is cut C and D are no longer connected.

The proof of relation (A6) follows the same lines as before provided that i and j are substituted by the sets C and D. Relations (A5) and (A6) are also valid for continuum percolation and directed percolation. In the case of site percolation p is the density of occupied sites and $\langle \lambda_{CD} \rangle$ is the average number of sites such that if one is removed C and D are no longer connected.

Relations (A5) and (A6) are trivially extended to the set of biconnected points (see equation (11)).

A particular case of equation (A6) is

$$p\ \mathrm{d}R/\mathrm{d}p = \langle \lambda \rangle \tag{A7}$$

where R is the probability that in a hypercube of linear dimension b one face is connected to the opposite one, and $\langle \lambda \rangle$ is the average number of cutting bonds such that if one is cut the two faces are not connected.

Finally we give another relation which can be proved in the same way as equation (A5):

$$p\ \mathrm{d}p_{ij}^{(\mathrm{f})}/\mathrm{d}p = \langle \lambda_{ij}^{(\mathrm{f})} \rangle - \langle \mu_{ij}^{\infty} \rangle \tag{A8}$$

where $p_{ij}^{(\mathrm{f})}$ is the probability that i and j are connected via a finite cluster, $\langle \lambda_{ij}^{(\mathrm{f})} \rangle$ is the average number of cutting bonds between i and j in the finite clusters and $\langle \mu_{ij}^{\infty} \rangle$ is the average number of bonds such that if one is cut i and j are both disconnected from the infinite cluster.

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